Physical Meaning of Complex Solution

Let us explain physical meaning of the complex turbulent solution. So, we will consider real solution of ordinary differential equations system \( x_\alpha(t) \).

Let us assume that initial data have an average value \( \langle x_\alpha \rangle \) and mean root square \( \langle |\Delta x_\alpha|^2 \rangle \).

Mean root square of initial data for Navier – Stokes equation is defined by surface roughness or by initial data which are not precisely defined. Then, for mean root square of the solution we have

\[
\langle |\Delta x|^2 \rangle = \langle |\Delta x|^2 \rangle = \langle x^2 \rangle - 2\langle x \rangle \langle x \rangle + \langle x \rangle^2 = \langle x^2 \rangle - \langle x \rangle^2.
\]

Then

\[
\langle x^2 \rangle = \langle x \rangle^2 + \langle |\Delta x|^2 \rangle = \langle x \rangle + i\sqrt{\langle |\Delta x|^2 \rangle} \cdot \alpha; \quad |\alpha| = 1,
\]

and the complex number \( \alpha \) is chosen in such a way that the imaginary part had positive or negative value. Mean square deviation satisfies this condition. But sometimes the mean square deviation is positive, for example, in case of dielectric permeability where positive and negative charges have an influence.

In this case we have a formula \( \varepsilon = \varepsilon_0 + \frac{4\pi\mu\sigma}{\omega} \), where real part is proportional to positive mean square dipole deviation and conductivity is proportional to average value. But conductivity is divided by frequency which has positive and negative sign.

Therefore, algorithm for finding of average solution or average solution in phase space and its mean root square is reduced to finding of complex solution. The average solution corresponds to real part of solution, and second power of complex part corresponds to mean root square of the solution. This is physical meaning of complex solution, real part is an average solution, and imaginary part is a mean square deviation. And real and imaginary parts are orthogonal and form complex space. Really, according to inverse Pythagoras theorem, due to formula (1) mathematical mean value and mean square deviation form legs and average square is a hypotenuse.

Here we would like to note that when calculating the flow motion and one term of a series is taken into account, it is necessary to take square root of imaginary part as forward velocity is calculated. The imaginary part corresponds to square root of oscillatory part of dimensionless velocity.

This situation is similar to calculation of deviation at random choice of forward or back step with probability 1/2 and the point position after \( N \) steps is defined by \( \sqrt{N} \).
Real and imaginary parts of the solution are located on different axes of complex space. But if you average imaginary dimensionless part, you will have
\[ \langle x'_i(t) \rangle + i\sqrt{\langle |\Delta x'_i(t)|^2 \rangle} \rightarrow \langle x'_i(t) \rangle + i\sqrt{\langle |\Delta x'_i(t)|^2 \rangle}. \]

And the solution is equal to the module of the last value and, for different roughness, the imaginary part of the solution should be multiplied by averaging multiplier. At the same time, if all coefficients of a series in non-linear equations system are calculated, it is not necessary to calculate square root of imaginary part. It is necessary to summarize complex values and to calculate module of the sum.

Now we will show that the imaginary part of complex derivative of coordinate in phase space of the differential equation forms pulsing coordinate motion in phase space, i.e. in space of variables
\[ \langle x'_i(t) \rangle + i\sqrt{\langle |\Delta x'_i(t)|^2 \rangle}. \]

Average values are used for variables as, at molecular level, the medium is not smooth.

**Lemma.** Complex solution yields fluctuating pulsing function of flow motion coordinates. The imaginary part of velocity corresponds to rotation speed in phase space. As rotation radius is known, it is also possible to determine rotation frequency. In the rotation plane, complex velocity with constant rotation radius and constant frequency can be written in the form
\[ V'_i + iV''_i = V_0 \exp(i\omega t). \]

In case of varying over the space stationary speed, locally, this formula can be written for one plane as
\[ V'_i(x,y) + iV''_i(x,y) = V_0(x,y) \exp \left[ \int_0^t \omega(x,y,u)du \right], \]
and frequency is dependent on time as the phase shift is provided as a result of harmonic oscillations in neighboring points. Sum of harmonic oscillations with different time-dependent frequencies defines pulsing mode in phase space at stationary complex velocity. That is, this complex velocity defines the coordinates of phase space points pulsing in time. The situation is similar to existence of several stationary vortices defining the pulsing rotation of the flow.

**Lemma 6.** Three-dimensional flow velocity can be written in the form
\[ V'_i = V'_0 + iV''_0 = V'_0 \exp(i\varphi_i); \quad \varphi_i = \arg(V'_0 + iV''_0). \]

And velocity is defined in the form of integral of tangent acceleration by formula
\[
V'_0 = \int \int \frac{\sum_{k=1}^{3} V'_k(u) V'_k(u)}{du} du + V'_0(t_0) = \\
= \int \int \frac{\sum_{k=1}^{3} [V'_{k}(u) + V''_{k}(u)]}{du} du + V'_0(t_0).
\]

Integral of perpendicular component of acceleration defines perpendicular component of velocity by formula
\[
V''_0 = \text{Im} V'(t_0) = \int w_{n}(u)du = \int \frac{\eta_i(u)|\text{Im} V|^2}{\rho(u)} du = \int |\text{Im} V(s)| \frac{\eta_i(s)}{\rho(s)} ds = \\
= \int \text{Im} V \int_0^t \text{Im} V dt_i = \int \text{Im} V |dt_i|, \quad |\text{Im} V| \neq \text{const}; \\
\sum_{k=1}^{3} [V'_{k}(u) + V''_{k}(u)] = |V'|^2; \\
dt_i(s) = \frac{\eta_i(s)ds}{\rho(s)}; \\
t_i(\tau) = \frac{\text{Im} V}{|\text{Im} V|}.
\]
At that value of local velocity is
$V_0' (\tau_0) = \text{Im} V \tau_0$, $V_0'' (\tau_0) = \text{Re} (V') \tau_0$. But value
of velocity obtained as a result of integration of
centripetal acceleration is not zero ($V_0'' (\tau) \neq 0$),
but this velocity become equal to zero for the
same initial point, at constant particle velocity
and constant curvature radius with rotation pe-
riod $T = \frac{2 \pi R}{V_0'}$ where $R$ – curvature radius. For
variable particle velocity depending on time,
when one of the integrals $\int_0^\tau t_0 |V| dt_0 = 0$, which, at
finite curvature radius of one sign of the trajec-
tory, is finite and equal to

$$T = \frac{1}{2} \int_0^\infty \frac{d\varphi}{\sqrt{V(\varphi)}} = \int_0^\infty \frac{dss}{\sqrt{V(s)}} + 2 \pi = \int_0^\infty \frac{dss}{R(s)}$$

as tangential direction $t_\varphi$ changes sign in the
course of rotation.

Tangential acceleration is defined by for-

$$w_t = d \left( \sum_{k=1}^3 \left[ (V_0')^2 + (V_0'')^2 \right] \right) / dt.$$

Direction of velocities $\Delta V_0', \Delta V_0''$ is orthog-
nonal, their sum yields increase of motion ve-
locity module

$$\sum_{i=1}^3 (2V_i)^2 = \sum_{i=1}^3 \left[ (V_0')^2 + (V_0'')^2 \right] = \sum_{i=1}^3 |V_0' + iV_0''|^2,$$

as

$$\sum_{i=1}^3 (w_t)^2 = \sum_{i=1}^3 \left[ (w_0')^2 + (w_0'')^2 \right].$$

Components of these projections, differenti-
able with respect to time, define tangen-
tial and orthogonal accelerations. At the same
time, concepts of tangential and orthogonal velo-
cities are entered which, in the Cartesian
space, are not orthogonal to $V_r, V_t$ but
in six-measured complex space they are or-
thogonal, and their module of complex vector
$V = V_0 + iV_0''$ is equal to

$$\sum_{i=1}^3 |V_i|^2 = \sum_{i=1}^3 \left[ (V_0')^2 + (V_0'')^2 \right] = \sum_{i=1}^3 |V_0' + iV_0''|^2.$$

It can be proved by use of expression

$$V_t = \sum_{i=1}^3 V_i \text{e}_a, \ V_{nt} = \sum_{i=1}^3 V_{nt} \text{e}_{nt}$$

and calculation of
module as product of complex conjugate vec-
tors taking into account orthogonality of six
real unit vectors.

**Conclusions**

Thus, solution of Navier – Stokes equations for not multiple balance po-
sitions is obtained. It is defined by expressions

$$V(t, r) = \sum_{n=1}^N x_n(t) \psi_n(r),$$

$$\sum_{s=1}^S \lambda_i \ln \left( x_i - a_i^l \right) \bigg|_{x_0} = H_i (t, t_0), \ l = 1, ..., 2N;$$

$$\lambda_i = \frac{1}{\left( a_i^l - a_i^{l-1} \right) \left( a_i^l - a_i^{l+1} \right) \left( a_i^l - a_i^{l-1} \right) \left( a_i^l - a_i^{l+1} \right)}$$

where values $a_i^l$ are coordinates of balance po-

Laminar solution corresponds to the solution of linear problem with convective term aver-
egating; structure of turbulent solution is

$$V(t, r) = \sum_{n=1}^N g_n(t) \psi_n(r) + a'$$

where $g_n(t)$ – known defined continuous func-
tion, value of $g_n(t) + a_0 \psi(t, x_0) + \pi n$ is de-

At that, the solution contains a lot of poles which, for real solution and real initial data,
yield infinity.

At real time and complex initial conditions
which define complex value of $g_n(t)$, and,
as $g(t)$ is real, the complex solution is

At that, formula

$$\sum_{s=1}^S \lambda_i \ln \left( x_i - a_i^l \right) \bigg|_{x_0} = H_i (t, t_0), \ l = 1, ..., N.$$ (2)

may have branching points in which the so-
lution continuously passes into other branch of
the solution. This does not contradict the
theorem of solution uniqueness for Cauchi
problem as the left part of the differen-
tial equation tends to infinity in branching
point. Derivative of right part of ordinary
differential equation also tends to infinity in
branching point. So we have a point of dis-
continuous solution. But this solution can be
continued by a formula (2).

This situation is similar to Schrödinger
equation when generally we have finite number
of solutions. It is not surprising as Schrödinger
equation can be reduced to Navier – Stokes
equation. Now we will prove it. For this we
will write down Schrödinger equation and will
transform it using equality

$$\frac{\partial^2 \psi}{\partial x^2} = -\psi \left[ \frac{\partial^2 \ln \psi}{\partial x^2} + \frac{1}{\psi^2} \left( \frac{\partial \psi}{\partial x} \right)^2 \right].$$
Dividing the equation by mass \( m \psi \) we obtain the equation

\[
\frac{i \hbar}{m} \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2m} \sum_{i=1}^{3} \frac{\partial^2 \psi}{\partial x_i^2} + U \psi = -\frac{\hbar^2}{2m} \psi \sum_{i=1}^{3} \left[ \frac{\partial^2 \ln \psi}{\partial x_i^2} + \frac{1}{\psi^2} \left( \frac{\partial \psi}{\partial x_i} \right)^2 \right] + U \psi.
\]

Now we will write a private derivative equation, will take a gradient of both parts of equation and will enter real velocity to the formula

\[
\mathbf{V} = \frac{-i \hbar}{m} \nabla \ln \psi,
\]

\[
\frac{\partial i}{m} \frac{\hbar}{m} \nabla \ln \psi + \frac{\hbar^2}{m^2} \sum_{i=1}^{3} \frac{\partial \ln \psi}{\partial x_i} \frac{\partial \nabla \ln \psi}{\partial x_i} + \frac{i \hbar}{m} \sum_{i=1}^{3} \frac{\partial^2 \ln \psi}{\partial x_i^2} + \frac{\nabla U}{m}.
\]

Substituting velocity value into transformed Schrödinger equation, we have

\[
\frac{\partial V}{\partial t} + \sum_{i=1}^{3} V_i \frac{\partial V}{\partial x_i} = v \sum_{i=1}^{3} \frac{\partial^2 V}{\partial x_i^2} - \frac{\partial U}{m}, \quad \mathbf{V} = \frac{i \hbar}{2m}.
\]

Now we have three-dimensional Navier – Stokes equation with pressure corresponding to potential. Nevertheless, the hydrodynamic problem differs from the equation of Navier – Stokes derived from Schrödinger equation and continuity equation.

At the same time it is possible to draw an analogy between laminar single-value mode and free, single-value description of bodies.

Between turbulent mode, having finite number of solutions, and description of bound particles having finite number of solutions. In case of turbulent complex and laminar real modes there is a boundary between them and critical Reynolds number. The similar boundary is available between free and bound particles description, which corresponds to energy transition from negative to positive state. In turn, Navier – Stokes equation has to have discrete energy levels of turbulent flow states, transitions between these states with energy emission or absorption have to be realized.

The boundary between free particles description and bound particles description can be defined, this is transition to complex quantum number or to infinity of the main quantum number of hydrogen atom. At that, infinite quantum number of hydrogen atom, passing through zero value of expression \( 1/n^2 \) where \( n \) – main quantum number, becomes imaginary and continuous. Wave function of free motion, which is continuous at continuous energy, corresponds to laminar solution of hydrodynamic problem for which single valued solution exists. And for large quantum number, the system is quasi-classical, i.e. for quantum number which is close to boundary (quantum number is equal to infinity) system is almost classical.

And there is a boundary between free solution and solution which describes bound states. This is zero energy value and, likewise non-linear private derivatives equations, boundary exists between turbulent complex solution and laminar real solution.